Non-linear Eigenmode Computations for Superconducting Cavities with a Surface Impedance Condition

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In this paper, the computation of quality factors, by exploiting surface impedance boundary conditions and non-linear eigenvalue solvers, is presented. In the last few years, the accurate calculation of losses and resonance frequencies of superconducting cavities has become a central part in the design of high-performance particle accelerators. However, solving London's equations in the superconductor is computationally impossible, since the material penetration depth is many orders of magnitude smaller than the resonator dimensions. Therefore, it is mandatory to simulate the behaviour of the material by taking advantage of a surface impedance boundary condition (SIBC). Unfortunately, when losses are taken into account in the superconducting material, the SIBC introduces a square-root operator in the corresponding eigenvalue formulation. Thus, the use of a non-linear eigenvalue solver is required. After reviewing some theoretical aspects, this paper presents a natural solution to couple a SIBC formulation with a non-linear eigenvalue solver. It finally concludes by discussing some numerical results.

***Index Terms*—Cavity resonator, non-linear eigenvalue problem, superconductivity, surface impedance boundary condition.**

I. Introduction

THE ACCURATE computation of resonance frequencies and quality factors of superconducting resonators is of paramount importance in particle accelerators design. This higher precision necessitates a more accurate description of superconducting screening currents than the classical perfect electric conductor (PEC) approximation. According to the two-fluid model coupled with London's equations [1], a superconducting material will introduce losses (even if they are small) and a frequency shift (compared to a PEC approximation) in the resonator. These two phenomena must be taken into account for two main reasons. First, the accelerating cavities have to be tuned with a high precision, requiring thus the accurate computation of eigenmodes. Second, the thermal load induced by the losses has the be accurately estimated.

II. Time-harmonic London's Equations

The electromagnetic behaviour of an ideal superconductor is described by the London equations. According to the first London equation, expressed in a time-harmonic framework, the electric field $\mathbf{e}$ and the electrical current density $\mathbf{j}$ are related by [1]:

$$\mathbf{j} = \left(\jmath\mu_0\lambda_L^2\Re(\omega)\right)^{-1}\mathbf{e}, \tag{1}$$

where $\jmath$ is the imaginary unit, μ_0 is the magnetic permeability in vacuum, λ_L is the London penetration depth and ω is the angular frequency[1]. By defining

$$\sigma_s = \left(\jmath\mu_0\lambda_L^2\Re(\omega)\right)^{-1}, \tag{2}$$

it becomes clear that (1) is nothing but a local circuit law for an inductor. Furthermore, it can be shown that the second London equation reduces to the time-harmonic Faraday law. Therefore, the classical electromagnetic theory can be used, albeit with a new definition of the conductivity. Let us finally notice that the real part operator is mandatory in a time-harmonic context. Indeed, since ω can be complex, its imaginary part will lead to nonphysical losses in (2) if the real part operator is omitted.

III. Two-fluid Model

The previous description of superconductivity is only valid at absolute zero temperature. For higher values[2], the two-fluid model suggests that two families of charge carriers exist in the material [1]: the super carriers (propagating without losses in the medium) and the normal carriers (responsible of the losses), their proportion being a function of the temperature. The existence of normal carriers can simply be taken into account by adding a real part to the conductivity [1]:

$$\sigma = \sigma_s + \sigma_n = \left(\jmath\mu_0\lambda_L^2\Re(\omega)\right)^{-1} + \sigma_n, \tag{3}$$

where σ_n is a real value which can be derived from the Drude model.

IV. Surface Impedance Boundary Condition

The penetration depth of a wave inside a (super)conducting material is extremely small (*e.g.* 2μm for copper for a plane wave at 1GHz and 39nm for superconducting niobium) compared to the size of the whole simulated structure (*e.g.* 103mm for a TESLA cavity). Therefore, a volume based numerical method requires a large number of mesh elements in order to resolve the penetration depth. Instead, a surface impedance boundary condition (SIBC) can be applied to avoid meshing the highly conductive parts. The central part of an SIBC is an operator S that relates (exactly or approximately) the magnetic and electric fields at the conducting interface. Formally, the SIBC reads:

$$\mathbf{n}\times\mathbf{h} = S(\mathbf{e}) \qquad \text{on } \partial\Omega, \tag{4}$$

[1] In this paper, the time-harmonic convention is $e^{\jmath\omega t}$, with t being the time.

[2] But below the critical temperature, where superconducting properties are completely lost.

where $\mathbf{h}$ is the magnetic field, S the SIBC operator, Ω the computational domain as depicted in Fig. 1, $\partial\Omega$ its boundary and $\mathbf{n}$ its outwardly oriented normal vector.

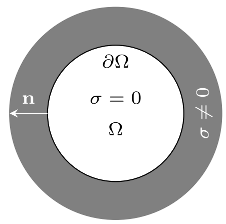

Fig. 1. Computational domain Ω and its boundary $\partial\Omega$, at which a SIBC models the highly conductive parts in grey.

In the case of a plane wave penetrating an infinite, flat and homogeneous conductor, the SIBC operator reads [2]:

$$S(\mathbf{e}) = \left(\sqrt{\frac{\jmath\mu_0\Re(\omega)}{\sigma}}\right)^{-1}\mathbf{e}, \qquad (5)$$

which is also known as the Leontovich SIBC. It is worth noticing that there exists higher-order operators taking the interface curvature into account [2]. In any case, the SIBC operator introduces a factor with the square root of the real part of the angular frequency. As a consequence, the resulting eigenvalue problem is non-linear. The major difference between a superconductor and a normal conductor is the complex-valued conductivity (3).

V. Non-linear Eigenvalue Solver

The weak formulation of the cavity eigenvalue problem reads [3]:

$$\begin{aligned}&\int_\Omega \mu^{-1}\mathbf{curl}\,\mathbf{e}\cdot\mathbf{curl}\,\mathbf{e}'\mathrm{d}\Omega - \omega^2\int_\Omega \varepsilon\mathbf{e}\cdot\mathbf{e}'\mathrm{d}\Omega \\ &-\jmath\omega\int_{\partial\Omega} S(\mathbf{e},\omega)\cdot\mathbf{e}'\mathrm{d}\partial\Omega = 0 \qquad \forall\mathbf{e}'\in H(\mathbf{curl}\,,\Omega), \qquad (6)\end{aligned}$$

where $\mathbf{e}'$ is a vectorial test function and $H(\mathbf{curl}\,,\Omega)$ the function space of square-integrable vector fields with square-integrable curls over Ω. Because of the SIBC, it can be directly noticed that (6) is a non-linear eigenvalue problem with eigenpairs $(\omega, \mathbf{e})$. Equation (6) is solved by the contour-integral method proposed in [4]. The algorithm samples (6) for a number of angular frequencies ω_i, chosen on a closed contour $\mathcal{C}$ in the complex plane. By combining the calculated solutions, a linear eigenvalue problem can be constructed, of which the eigenvalues approximate the eigenvalues of (6) enclosed by $\mathcal{C}$.

VI. Validation

The algorithm and its implementation are validated by calculating the quality factor of the fundamental mode of a spherical cavity with a radius of 100mm. The conductivity of the cavity walls is ranging between $10^{15}\Omega^{-1}\mathrm{m}^{-1}$ and $10^{0}\Omega^{-1}\mathrm{m}^{-1}$. The problem is discretized with the finite element (FE) method and Nédélec edge elements on a first-order tetrahedral mesh with a uniform density of 10 mesh elements per radius. The Leontovich operator (5) is used to model the conducting walls.

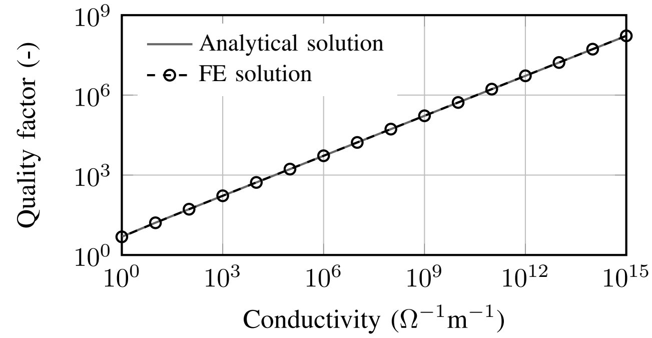

Fig. 2. Quality factor of a spherical cavity with non-perfectly conducting walls.

The computed results are then compared to the analytical solution developed in [5], as depicted in Fig. 2. It is obvious that even for relatively small conductivities, the FE model exhibits an excellent accuracy, with a relative error of the order of 0.2% in the range $[10^{15}, 10^{3}]\Omega^{-1}\mathrm{m}^{-1}$. It is worth noticing that since we use a straight mesh, the spherical geometry is not well approximated, which limits the accuracy.

For the same cavity but made of superconducting niobium and operated at $4K$, a quality factor of $Q = 2.33\times10^9$ for the fundamental mode is found. By using a complex conductivity in the analytical solution of [5], the FE result only deviates from 0.2% from the reference solution.

VII. Summary

This paper describes how conductors, with normal and superconducting carriers, can be taken into account in eigenvalue problems by exploiting a SIBC. This operator leads to a non-linear eigenvalue problem, which is then solved by a contour integral method. The overall algorithm is able to calculate lossy eigenmodes up to a high accuracy.

Acknowledgements

This research is funded by the German Bundesministerium für Bildung und Forschung (www.bmbf.de) as the project BMBF-05P15RDRBB "Ultra-Sensitive Strahlstrommessung für zukünftige Beschleunigeranlagen".

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